Discrete Mathematics - Second Midterm Exam Warm-up

1. Consider the graph G: (i) $K_{3,4}$, (ii) C_6, (iii) Petersen graph.

   a) What is the number of vertices and the number of edges of G?

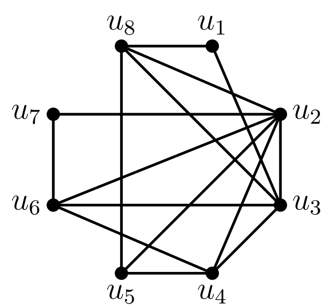
   b) Write down the degree sequence of G.

   c) Determine the adjacency matrix of G.

   d) Is this graph regular? Explain shortly why.

   e) Is this graph bipartite? If yes, write down the vertex classes. If no, explain shortly why.

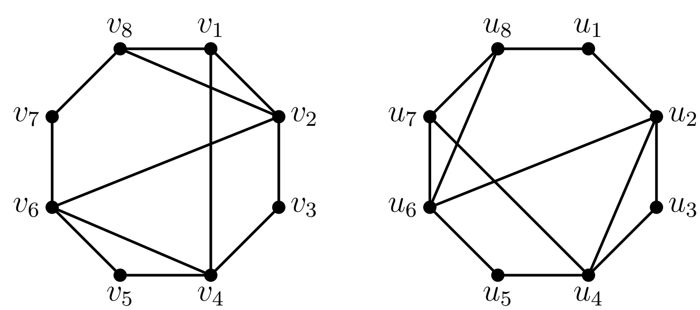
2. For a graph G pictured below draw the following subgraphs:
   (i) $G[\{u_1, u_3, u_5, u_6, u_8\}]$, (ii) $G - \{u_1, u_2, u_5\}$, (iii) $G[\{u_1u_3, u_1u_8, u_2u_5, u_2u_6, u_5u_8\}]$.



3. Draw all nonisomorphic graphs with 6 vertices and 4 edges.

4. Are these graphs isomorphic? If yes, write down an isomorphism. If no, explain why.

a)



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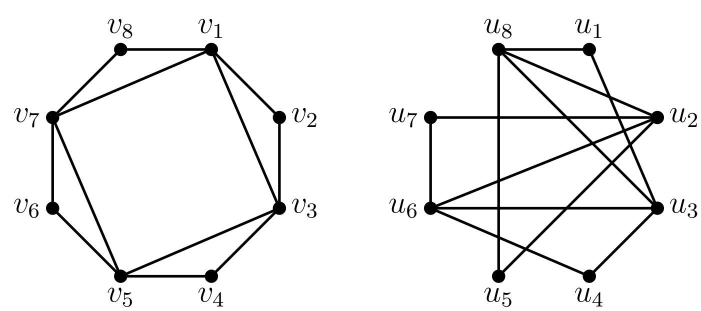
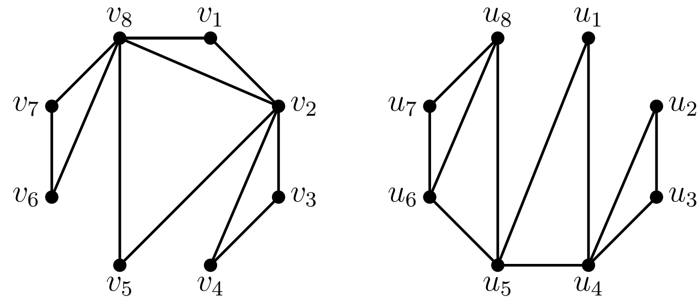
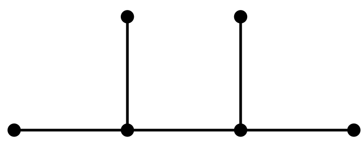


c)



5. Determine the number of different labelings of vertices of graph G by numbers $\{1, 2, \ldots, v_G\}$.

a)



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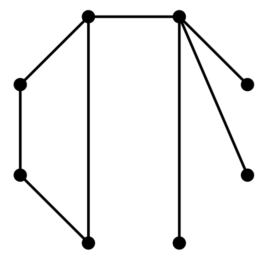

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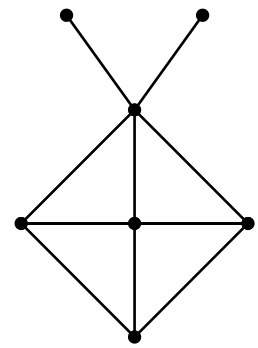
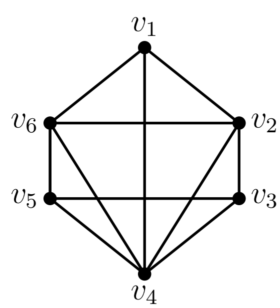


6. Find the number of walks of length 4 between v_1 and v_2 in the graph given below.



7. Let $A_{t\times t}$ be a $t\times t$ matrix with 0's on the main diagonal and 1's beside that. Find the number of connected components of the graph G given by the following adjacency matrix. What is the structure of graph G?

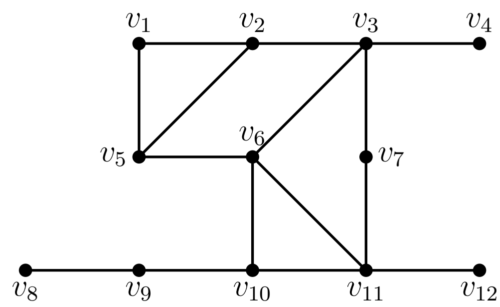
$$A(G) = \begin{bmatrix} A_{k\times k} & 0 & 0 \\ 0 & A_{n\times n} & 0 \\ 0 & 0 & A_{m\times m} \end{bmatrix}$$

8. Find the number of triangles (cycles C_3) in K_6 and the number of walks of length 3 between two distinct vertices of K_6.

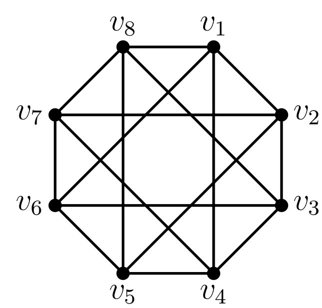
9. Find the adjacency matrix and the incidence matrix of the path P_7.

10. Determine $\kappa(G)$ and $\lambda(G)$, where G is (i) W_{10}, (ii) the Petersen graph, (iii) K_6.

11. Find all cut vertices and all cut edges of the graph given below:



12. Determine whether given graph has an Euler tour or an Euler walk. If yes, use Fleury's algorithm to find it. If no, explain why.



13. Determine, whether the graph W_{100} has a Hamilton path/cycle. If yes, find such a path. If no, explain why.

14. Does a graph with degree sequence (i) $(3, 3, 3, 3, 3, 3, 3, 3)$, (ii) $(6, 3, 3, 3, 3, 2, 2)$ have an Euler tour? Explain your answer.

15. Which of the graphs (i) $K_{n,n}$, (ii) $K_{n,n+1}$, (iii) $K_{n,2n}$ contains a Hamilton cycle or a Hamilton path? If the graph contains a Hamilton cycle, can you prove it using Ore's theorem? How about Dirac's theorem?

Hints and solutions

1.(i)

a) $v(K_{3,4}) = 7$, $e(K_{3,4}) = 12$

b) $(3, 3, 3, 3, 4, 4, 4)$

c)

$$A(K_{3,4}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

d) no

e) yes

1.(ii)

a) $v(C_6) = e(C_6) = 6$

b) $(2, 2, 2, 2, 2, 2)$

c)

$$A(C_6) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

d) yes

e) yes

1.(iii) G - Petersen graph

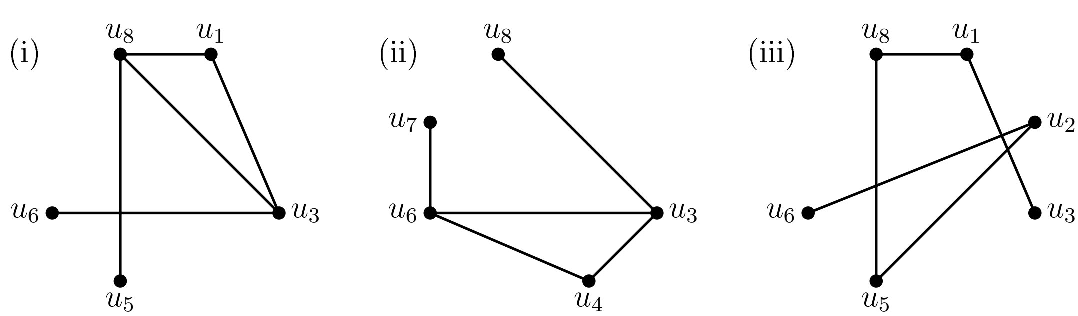
a) $v(G) = 10$, $e(G) = 15$

b) $(3, 3, 3, 3, 3, 3, 3, 3, 3, 3)$

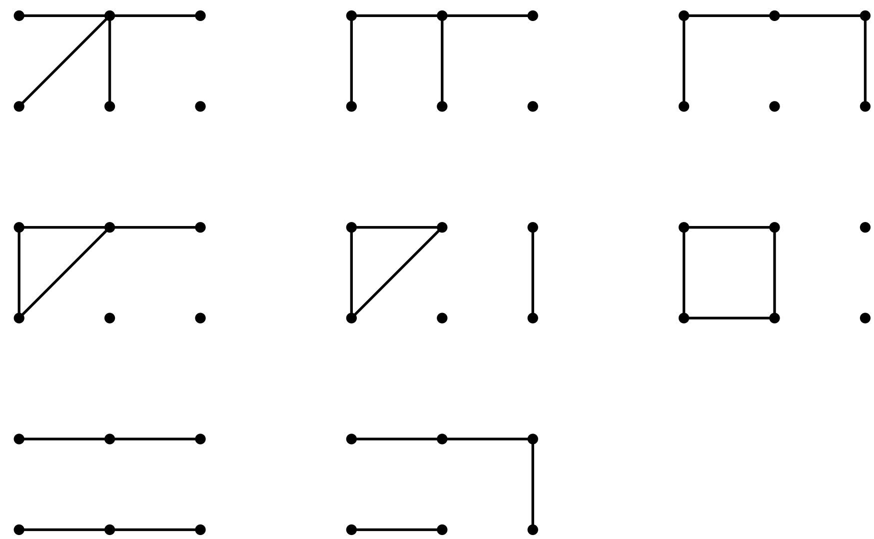
d) yes

e) no

2.



3.



4. a)

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$\varphi(v)$	u_8	u_5	u_2	u_7	u_6	u_4	u_3	u_1

b)

v	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$\varphi(v)$	u_7	u_6	u_5	u_4	u_3	u_2	u_1	u_8

c) no isomorphism - there are two paths of length two between vertices of degree 5 in the graph on the left, and only one such path in the graph on the right

5. a) $\binom{6}{2} \cdot \binom{4}{2} = 90$, b) $8 \cdot 7 \cdot \binom{6}{3} \cdot 3 = 3360$, c) $7 \cdot 6 \cdot 5 \cdot \binom{4}{2} = 1260$

6. There are $3 \cdot 2 + 2 \cdot (4 + 1 + 3 + 3 + 2) = 32$ such walks since

$$A^2(G) = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 1 & 3 & 3 & 2 \\ 2 & 1 & 3 & 2 & 1 & 3 \\ 2 & 3 & 2 & 5 & 2 & 3 \\ 2 & 3 & 1 & 2 & 3 & 1 \\ 2 & 2 & 3 & 3 & 1 & 4 \end{bmatrix}$$

7. Three connected components isomorphic, respectively, to K_k, K_n and K_m.

8. 20 triangles, 21 walks of length 3 between any two distinct vertices

9. $$M(P_7) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad A(P_7) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

10. (i) $\kappa(W_{10}) = 3, \quad \lambda(W_{10}) = 3$

(ii) G - Petersen graph, $\kappa(G) = 3, \quad \lambda(G) = 3$

(iii) $\kappa(K_6)$ - not defined, $\lambda(K_6) = 5$

11. cut vertices: v_3, v_9, v_{10}, v_{11}, cut edges: $v_3v_4, v_8v_9, v_9v_{10}, v_{11}v_{12}$

12. Euler tour: $v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_1v_6, v_6v_3, v_3v_8, v_8v_5, v_5v_2, v_2v_7, v_7v_4, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_1$

13. W_{100} contains a Hamilton cycle - start from the middle vertex, then go around the wheel and back to the middle vertex

14. (i) doesn't have two, take a graph consisting of two copies of K_4, (ii) no, four vertices of odd degree

15. $K_{n,n}$ contains a Hamilton cycle (one can use both Ore's theorem and Dirac's theorem), $K_{n+1,n}$ contains a Hamilton path, $K_{n,2n}$ contains a Hamilton path only for $n = 1$